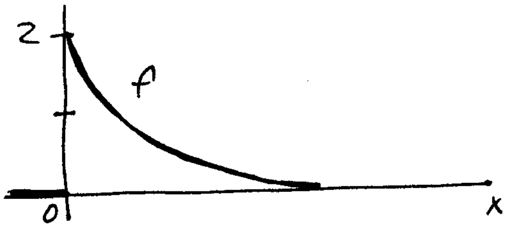
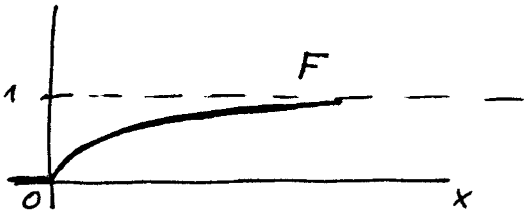
17.6.03

(1) $f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

a) spočítat F + grafy

$$F = \int_{-\infty}^{x} f(t)\,dt = \int_0^x 2e^{-2t}\,dt = 2\left[\frac{e^{-2t}}{-2}\right]_0^x = -\frac{e^{-2x}}{1} + 1 = 1 - e^{-2x}$$



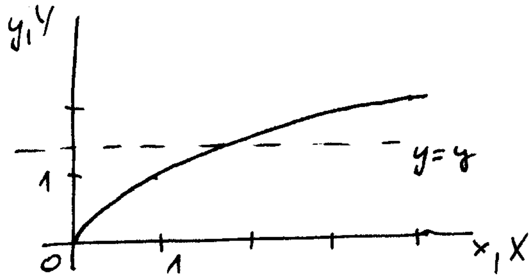


b) $0{,}1 = P(X \geq a) = 1 - P(X \leq a) = 1 - F(a) \Rightarrow F(a) = 0{,}9$

$$1 - e^{-2a} = 0{,}9 \Rightarrow e^{-2a} = 0{,}1 \Rightarrow -2a = \ln(0{,}1)$$

$$\underline{a = \tfrac{1}{2}\ln 10} \doteq 1{,}151$$

c) $Y = \sqrt{X}$, spočítat hustotu



$x \in (0,\infty)$, $y \in (0,\infty)$

$\underline{y \geq 0}$: $G(y) = P(Y \leq y) = P(\sqrt{X} \leq y) =$

$= P(X \leq y^2) = F(y^2)$

$G(Y) = 1 - e^{-2y^2}$

$\underline{g(y)} = G'(y) = \underline{4y\,e^{-2y^2}}$, $\underline{y > 0}$

d) $E(X^2 + Y^2) = E(X^2 + X) = E(X^2) + E(X) = \frac{1}{2} + \frac{1}{2} = \underline{1}$

$$E(X) = \int_0^{\infty} 2x e^{-2x}\,dx = 2\left[-\frac{1}{2}x e^{-2x} - \frac{1}{4}e^{-2x}\right]_0^{\infty} = 2\left(-0 - 0 + 0 + \frac{1}{4}\right) = \underline{\frac{1}{2}}$$

$$E(X^2) = \int_0^{\infty} 2x^2 e^{-2x}\,dx = 2\left[-\frac{1}{2}x^2 e^{-2x}\right]_0^{\infty} + 2\int_0^{\infty} x e^{-2x}\,dx =$$

$$= \left[-x^2 e^{-2x}\right]_0^{\infty} + \frac{1}{2} = -0 + 0 + \frac{1}{2} = \underline{\frac{1}{2}}$$

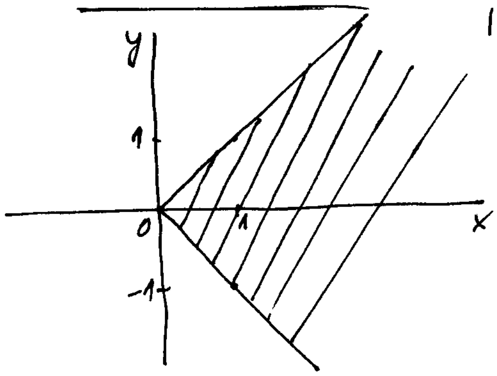
② X, Y nezávislé, rozdělení $N(0,1)$

$$N(0,1) \Rightarrow \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

X, Y nezávislé $\Rightarrow$ sdružená hustota je rovna součinu hustot marginálních:

$$f(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$$

a) $$P(|Y| \le X) = \iint_{|y| \le x} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} \, dx\, dy = \left| \begin{array}{l} \text{polár.} \\ 0 < \rho < \infty \\ -\pi/4 \le \varphi \le \pi/4 \end{array} \right| =$$



$$= \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} \left(\int_0^\infty e^{-\frac{\rho^2}{2}} \rho \, d\rho \right) d\varphi = \left| \begin{array}{l} \text{sub.: } \frac{\rho^2}{2} = t \\ \rho\, d\rho = dt \end{array} \right| =$$

$$= \frac{1}{2\pi} \cdot \frac{\pi}{2} \left[-e^{-t} \right]_0^\infty = \frac{1}{4}(-0+1) = \underline{\underline{\frac{1}{4}}}$$

b) $$E(\pi(X^2+Y^2)) = \iint_{R^2} \pi(x^2+y^2) f(x,y)\, dx\, dy = \iint_{R^2} \pi(x^2+y^2) \cdot \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dx\, dy =$$

$$\overset{\text{do polár.}}{=} \int_0^{2\pi} \left(\int_0^\infty \frac{\rho^2}{2} \cdot e^{-\frac{\rho^2}{2}} \rho\, d\rho \right) d\varphi = \left| \begin{array}{l} \frac{\rho^2}{2} = t \\ \rho\, d\rho = dt \end{array} \right| = 2\pi \int_0^\infty t \cdot e^{-t} dt = \left| \begin{array}{ll} u = t & v' = e^{-t} \\ u' = 1 & v = -e^{-t} \end{array} \right| =$$

$$= 2\pi \left[-t e^{-t} - e^{-t} \right]_0^\infty = 2\pi(-0-0+0+1) = \underline{\underline{2\pi}}$$

c) Hustota $X|1$ ... ?

$$f_1(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

③ $N(3, \sigma^2)$ a $P(|X-3| < 5) = 0{,}95$

$$P(-2 \leq X \leq 8) = F(8) - F(-2) = \Phi\left(\frac{8-3}{\sigma}\right) - \Phi\left(\frac{-2-3}{\sigma}\right) =$$

$$= \Phi\left(\frac{5}{\sigma}\right) - 1 + \Phi\left(\frac{5}{\sigma}\right) = 2\Phi\left(\frac{5}{\sigma}\right) - 1 = 0{,}95$$

$$\Phi\left(\frac{5}{\sigma}\right) = \frac{1{,}95}{2} = 0{,}975$$

$$\frac{5}{\sigma} = u_{0{,}975} \Rightarrow \sigma = \frac{5}{u_{0{,}975}} = \frac{5}{1{,}96}$$

$$\underline{\sigma = 2{,}55}$$

a) $\underline{P(-2{,}5 < X < 8{,}5)} = F(8{,}5) - F(-2{,}5) = \Phi\left(\frac{8{,}5-3}{2{,}55}\right) - \Phi\left(\frac{-2{,}5-3}{2{,}55}\right) =$

$$= 2\Phi\left(\frac{5{,}5}{2{,}55}\right) - 1 = 2\Phi(2{,}157) - 1 = 2 \cdot 0{,}984 - 1 = \underline{0{,}968}$$

b) $\underline{E(X^2 + 2X - 1)} = \cancel{*} = \underline{\sigma^2 + \mu^2 + 2\mu - 1}$

$$= E(X^2) + 2 \cdot E(X) - 1$$

$\underline{E(X)} = \underline{\mu}$

$D(X) = E(X^2) - (E(X))^2 = \sigma^2$

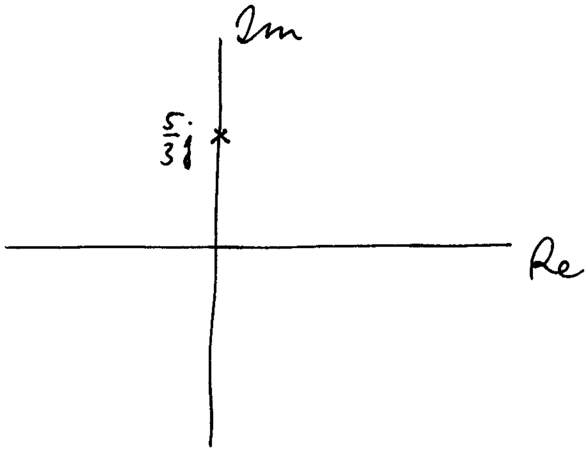
$\underline{E(X^2)} = D(X) + (E(X))^2 = \underline{\sigma^2 + \mu^2}$

(4) $\underline{tg\left(\frac{\pi}{2} + j\ln 2\right)}$

$$tg\, z = \frac{\sin z}{\cos z} = \frac{\frac{1}{2j}\left(e^{jz} - e^{-jz}\right)}{\frac{1}{2}\left(e^{jz} + e^{-jz}\right)} = \frac{e^{2jz} - 1}{j\left(e^{2jz} + 1\right)}$$

$$tg\left(\frac{\pi}{2} + j\cdot\ln 2\right) = \frac{e^{2j\left(\frac{\pi}{2} + j\ln 2\right)} - 1}{j\left(e^{2j\left(\frac{\pi}{2} + j\ln 2\right)} + 1\right)} = \frac{e^{j\pi - \ln 4} - 1}{j\left(e^{j\pi - \ln 4} + 1\right)} =$$

$$= \frac{-\frac{1}{4} - 1}{j\left(-\frac{1}{4} + 1\right)} = j\,\frac{\frac{5}{4}}{\frac{3}{4}} = \underline{\underline{\frac{5}{3}\,j}}$$



5 TEORIE

a) Sdruž. harm. funkce, jejich vztah k funkci holomorfní

Komplexní funkce: $f(z) = u(z) + j \cdot v(z)$

u – reálná, v – imaginární } část komplex. funkce f

Harmonické funkce: $\Delta u = 0$ & $\Delta v = 0$

(nutná podmínka holomorfnosti)

$f(z) = u(z) + j \cdot v(z)$ je holomorfní v oblasti $G \iff$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\underline{u}$ určuje $\underline{v}$ a $\underline{v}$ určuje $\underline{u}$.

b) Zadáno $E(x)$, $E(x^2)$, $P(|X+1| < \varepsilon) \geq 0{,}96$

Spočítat ε.

Čebyševova nerovnost: je-li X náhod. vel., která má konečnou stř. hodnotu a rozptyl, pak

$$\boxed{P(|X - E(X)| < \varepsilon) \geq 1 - \frac{D(X)}{\varepsilon^2}}$$

$$1 - \frac{D(X)}{\varepsilon^2} = 0{,}96$$

$$\frac{D(X)}{\varepsilon^2} = 0{,}04$$

$$\varepsilon^2 = \frac{D(X)}{0{,}04} = \frac{E(X^2) - (E(X))^2}{0{,}04}$$

$$\varepsilon = \frac{\sqrt{E(X^2) - (E(X))^2}}{0{,}2}$$